Problem Set 10
Optical Waveguides and Fibers (OWF)

Problem 1: Loss mechanisms in a standard single-mode fiber.

Consider a fiber for which the attenuation coefficient $\alpha(z)$ varies along z. The evolution of the power is given by the differential equation $\frac{\partial P(z)}{\partial z} = -\alpha(z)P(z)$.

a) Solve this differential equation for a general profile $\alpha(z)$?

Solution: Rearanging the differential equation we get $\frac{\partial P(z)}{P(z)} = -\alpha(z)\partial z$. Integrating both sides of this equation we get $\int_{P(0)}^{P(z)} \frac{dP}{P} = -\int_0^z \alpha(z')\,\mathrm{d}z'$, which finally leads to:

$$P(z) = P_0 e^{-\int_0^z \alpha(z')\,\mathrm{d}z'}, \tag{1}$$

where $P_0 \equiv P(0)$.

b) In most practical cases, fiber losses are given in dB/km. How is this value related to $\alpha(z)$?

Solution: The total attenuation of the fiber is

$$a[\mathrm{dB}] = 10\log_{10}\frac{P_0}{P(z)} = 10\log_{10}\frac{1}{e^{-\int_0^z \alpha(z')\,\mathrm{d}z'}} = \int_0^z \alpha(z')\,\mathrm{d}z' \cdot 10\log_{10} e \approx 4.34\int_0^z \alpha(z')\,\mathrm{d}z'. \tag{2}$$

In order to get the attenuation in dB/km, we need to divide a[dB] by z[km]. We finally have

$$\frac{a}{z}\left[\frac{\mathrm{dB}}{\mathrm{km}}\right] = \frac{4.34\int_0^z \alpha(z')\,\mathrm{d}z'}{z}. \tag{3}$$

c) Desribe the different loss-mechanisms in a standard single-mode fiber.

Solution: See lecture notes, section 5.6.1 “Fiber attenuation”.

Problem 2: Bit-rate limitation caused by dispersion and fiber length.

Consider a Gaussian pulse of inital temporal variance $\sigma_t(0)$, chirp parameter[1] α that propagates along a fiber with chromatic dispersion $\beta_c^{(2)} = -C_\lambda \frac{\lambda^2}{2\pi c}$. At each position z the complex envelope of the pulse can be written as $\underline{A}(z,t) = \frac{A_0}{\sqrt{Q(z)}}\exp\left(-\frac{(1-\mathrm{j}\alpha)t^2}{2\sigma_t^2(0)Q(z)}\right)$, where $Q(z) = 1 + (\mathrm{j}+\alpha)\frac{\beta_c^{(2)} z}{\sigma_t^2(0)}$. The evolution of the pulse width $\sigma_t(z)$ during the propagation along z is hence given by

$$\sigma_t(z) = \sigma_t(0)\sqrt{\left(1+\alpha\frac{\beta_c^{(2)} z}{\sigma_t^2(0)}\right)^2 + \left(\frac{\beta_c^{(2)} z}{\sigma_t^2(0)}\right)^2}. \tag{4}$$

Assume that you have a transmitter that can produce only unchirped ($\alpha = 0$) Gaussian pulses for on-off keying data signals. Let’s call B the symbol rate of the data stream and assume that two neighboring symbols can be distinguished only if the final temporal variance $\sigma_t(L)$ is smaller than half the bit duration, i.e., if $\sigma_t(L) < 1/(2B)$.

a) Sketch the behavior of $\sigma_t(L)$ vs. $\sigma_t(0)$.
What happens to $\sigma_t(L)$ if $\sigma_t(0)$ is made infinitely small?

[1] Note that in problem 2 α denotes the chirp parameter rather than the fiber loss coefficient as in Problem 1.

Solution: When using the unchirped pulse, the pulse width $\sigma_t(L)$ after length L is calculated with $\sigma_t(L) = \sigma_t(0)\sqrt{1+\left(\frac{\beta_c^{(2)}L}{\sigma_t^2(0)}\right)^2}$. For very small $\sigma_t(0)$ we get

$$\lim_{\sigma_t(0)\to 0}\sigma_t(L) = \lim_{\sigma_t(0)\to 0}\sigma_t(0)\sqrt{1+\left(\frac{\beta_c^{(2)}L}{\sigma_t^2(0)}\right)^2} = \infty,$$

and for very large $\sigma_t(0)$ we get the same; $\lim_{\sigma_t(0)\to\infty}\sigma_t(L) = \infty$. Therefore, there must be a minimum inbetween. See Fig. 1 for a sketch of the relation.

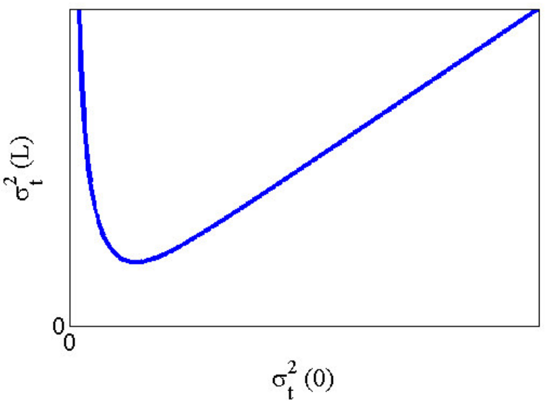

Figure 1: Qualitative relation between $\sigma_t(0)$ and $\sigma_t(L)$

b) There is an optimum initial time variance $\sigma_{t,\min}(0)$ for which the output pulse has a minimum width. Give a physical explanation of this observation by considering the extreme cases of a very long ($\sigma_t(0)\to\infty$) and a very short ($\sigma_t(0)\to 0$) input pulse. Calculate $\sigma_{t,\min}(0)$ in terms of L and $\beta_c^{(2)}$. What is the corresponding time variance $\sigma_{t,\min}(L)$ at the fiber output?

Solution: To find the optimum initial time variance we can calculate

$$\frac{\mathrm{d}}{\mathrm{d}\sigma_t(0)}\left(\sqrt{\sigma_t(0)^2+\frac{\left(\beta_c^{(2)}L\right)^2}{\sigma_t(0)^2}}\right) = 0$$

$$\frac{1}{\sqrt{\sigma_t(0)^2+\frac{\left(\beta_c^{(2)}L\right)^2}{\sigma_t(0)^2}}}\left(2\sigma_t(0)-2\frac{\left(\beta_c^{(2)}L\right)^2}{\sigma_t(0)^3}\right) = 0$$

$$\left(\frac{2\sigma_t(0)^4}{\sigma_t(0)^3}-2\frac{\left(\beta_c^{(2)}L\right)^2}{\sigma_t(0)^3}\right) = 0$$

$$\sigma_t(0)^4-\left(\beta_c^{(2)}L\right)^2 = 0$$

$$\sigma_{t,min}(0) = \sqrt{\left|\beta_c^{(2)}\right|L}$$

$$\Longrightarrow \quad \sigma_{t,\min}(L) = \sqrt{\left|\beta_c^{(2)}\right|L+\frac{\left(\beta_c^{(2)}L\right)^2}{\left|\beta_c^{(2)}\right|L}} = \sqrt{2\left|\beta_c^{(2)}\right|L}$$

c) What is the maximum symbol rate that can be transmitted through a fiber of given length L and dispersion $\beta_c^{(2)}$?

Solution: The condition $\sigma_{t,min}(L) = 1/(2B_{max})$ gives the maximum bandwidth, with the result from b) we can write

$$B_{max} = \frac{1}{2\sigma_{t,min}(L)} = \frac{1}{2\sqrt{2\left|\beta_c^{(2)}\right| L}}.$$

d) Assume now that the fiber has chromatic dispersion $C_\lambda = 16\,\frac{\text{ps}}{\text{nm}\cdot\text{km}}$. What is the maximum transmission length for symbol rates of $B = 10\,\text{GBd}$ and $B = 40\,\text{GBd}$? Remember that $\beta_c^{(2)} = -\frac{\lambda}{\omega}C_\lambda$.

Solution: From c) we can write

$$B_{max} = \frac{1}{2\sigma_{t,min}(L)} = \frac{1}{2\sqrt{2\left|\beta_c^{(2)}\right| L_{max}}}$$

$$L_{max} = \frac{1}{8\left|\beta_c^{(2)}\right| B_{max}^2}$$

$$L_{max} = \frac{1}{8\left|-C_\lambda \frac{\lambda^2}{2\pi c}\right| B_{max}^2}$$

with $\beta_c^{(2)} = -C_\lambda \frac{\lambda^2}{2\pi c} = -2.04 \cdot 10^{-26}\,\frac{\text{s}^2}{\text{m}}$ one can calculate the values for the different symbol rate.

$$L_{max,10G} = \frac{1}{8 \cdot 2.04 \cdot 10^{-26}\,\frac{\text{s}^2}{\text{m}}(10\text{GBd})^2} \approx 61.3\text{km}$$

$$L_{max,40G} = \frac{1}{8 \cdot 2.04 \cdot 10^{-26}\,\frac{\text{s}^2}{\text{m}}(40\text{GBd})^2} \approx 3.8\text{km}$$

Questions and Comments:

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